

AN EFFICIENT FAMILY OF OPTIMAL EIGHT-ORDER ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS

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Abstract. The prime objective of this paper is to design a new family of eighth-order iterative methods by accelerating the order of convergence and efficiency index of well existing seventh-order iterative method of [1] without using more function evaluations for finding simple roots of nonlinear equations. The presented iterative family requires three function and one derivative evaluations and thus agrees with the conjecture of Kung-Traub for the case $n = 4$ (i.e. optimal). We have also discussed the derivative free version of the proposed scheme. Numerical comparisons have been carried out to demonstrate the efficiency and the performances of proposed method.

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1. INTRODUCTION

The Newton's iterative method is one of the prominent methods for finding roots of a nonlinear equation

$$f(x) = 0.$$

It is well known that the order of convergence of the Newton's method is two. In real life problems, the evaluation of derivatives is difficult (sometimes not possible) or takes up a very long computational time, in that case it is hard to implement Newton method. To overcome this problem, Steffensen has provided an iterative method

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \quad (1.1)$$

with two function evaluations and the same convergence rate as Newton-Raphson's method. Solving nonlinear equations is one of the most important and gripping task in numerical analysis. The vast literature

is available on the solution of nonlinear equations or system of nonlinear equations, one may refer [4]-[9]. Very recently, in [10] Petkovic et al. have provided detail discussion on multipoint methods for solving nonlinear equations. Such type of schemes have drawn the attention of many researchers. In recent past, many researchers have focused to optimize the existing methods without evaluating additional functions and first derivative of functions.

Recently, Soleymani et al. has established seventh-order method defined in [1] is given by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f[x_n, y_n]} \cdot G(t_n) \\ x_{n+1} &= z_n - \frac{f(z_n)}{f[y_n, z_n]} \cdot H(t_n), \end{aligned} \quad (1.2)$$

where $t_n = \frac{f(y_n)}{f(x_n)}$ and $G(0) = G'(0) = 1$, $|G''(0)| \leq +\infty$; $H(0) = 1$, $H'(0) = 0$, $H''(0) = 2$, $|H^{(3)}(0)| \leq +\infty$.

To compare efficiency of different iterative methods the efficiency index is defined in [18, 19] and given by $p^{1/n}$, where p is the order of convergence and n be the number of function evaluations of the iterative method. Kung and Traub [8] presented a hypothesis on the optimality of the iterative methods by giving 2^{n-1} as the optimal order. Thus the efficiency index of the method (1.2) is $7^{1/4} \approx 1.626$ and clearly it is not optimal (because this method requires four function evaluations (three functions and one derivative) so for optimal its order of convergence should be $2^3 = 8$). The motive of this paper is to acceleraerate the order of convergence of the method (1.2) from seven to eight without adding more evaluations, and thus it will agrees with Kung-Traub conjecture as well as give higher efficiency index.

The rest of the paper is organized as follows: in section 2, we propose a new optimal eight-order iterative method for finding simple roots of nonlinear equations . Particular case of proposed iterative family have also been given. In section 3 an approach has been given to make our proposed method derivative free . In section 4, we employ some numerical examples to compare the performance of our new method with some existing eight-order methods. Finally, in the last section we furnished the concluding remarks and future work.

2. IMPROVED SCHEME AND CONVERGENCE ANALYSIS

In, this section, the order of convergence of the method (1.2) will be accelerated from seven to eight to make it optimal. The order of convergence of the method (1.2) is seven by using four function $[f(x_n), f'(x_n), f(y_n), f(z_n)]$ evaluations, which is clearly not optimal. To build an optimal eight-order method family of iterative methods without using more evaluations of the functions, we consider the following family

$$\begin{aligned} y_n &= x_n - A(t_1) \cdot \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - B(t_2) \cdot \frac{f(y_n)}{f[x_n, y_n]} \\ x_{n+1} &= z_n - \{P(t_2) + Q(t_3) + R(t_4)\} \cdot \frac{f(z_n)}{f[y_n, z_n]}, \end{aligned} \quad (2.1)$$

where $t_1 = \frac{f(x_n)}{f'(x_n)}$, $t_2 = \frac{f(y_n)}{f(x_n)}$, $t_3 = \frac{f(z_n)}{f(y_n)}$ and $t_4 = \frac{f(z_n)}{f(x_n)}$. The weight functions should be chosen such that the order arrives at optimal level eight. Theorem (2.1) gives the conditions on weight functions to reach at optimal level of convergence.

Theorem 2.1. *Let the function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ have sufficient number of continuous derivatives in a neighborhood D of simple root α of f . Then the method defined by (2.1) has eighth-order convergence, when the weight functions $A(t_1)$, $B(t_2)$, $P(t_2)$, $Q(t_3)$ and $R(t_4)$, satisfy the following conditions:*

$$\begin{aligned} A(0) &= 1, \quad A'(0) = 0, \quad A''(0) = 0, \quad |A^{(3)}(0)| \leq +\infty, \\ B(0) &= 1, \quad B'(0) = 1, \quad |B^{(3)}(1)| \leq +\infty, \\ R(0) &= 1 - P(0) - Q(0), \\ P'(0) &= 0, \quad P''(0) = 2, \quad P^{(3)}(0) = 6B''(0) - 12, \quad |P^{(4)}(0)| \leq +\infty, \\ Q'(0) &= 0, \quad |Q''(0)| \leq +\infty, \\ R'(0) &= 2, \quad |R''(0)| \leq +\infty. \end{aligned} \quad (2.2)$$

Proof. Let $e_n = x_n - \alpha$ be the error in the n^{th} iterate and $c_h = \frac{f^{(h)}(\alpha)}{h!}$, $h = 1, 2, 3, \dots$. We provide the Taylor's series expansion of each term involved in (2.1). By Taylor expansion around the simple root in the n^{th} iteration, we have

$$f(x_n) = c_1 e_n + c_2 e_n^2 + \dots + O(e_n^{10}), \quad (2.3)$$

and

$$f'(x_n) = c_1 + 2c_2e_n + \dots + O(e_n^9). \quad (2.4)$$

Furthermore, it can be easily find

$$\frac{f(x_n)}{f'(x_n)} = e_n - \frac{c_2e_n^2}{c_1} + \dots + O(e_n^9). \quad (2.5)$$

By considering this relation and $A(0) = 1$ we obtain

$$\begin{aligned} y_n &= \alpha + \left\{ \frac{c_2}{c_1} - A'(0) \right\} e_n^2 + \left\{ -\frac{2c_2^2}{c_1^2} + \frac{2(c_3 + c_2A'(0))}{c_1} - \frac{A''(0)}{2} \right\} e_n^3 \\ &\quad + \dots + O(e_n^9). \end{aligned} \quad (2.6)$$

At this time, we should expand $f'(y_n)$ around the root by taking into consideration (2.6). Accordingly, we have

$$\begin{aligned} f(y_n) &= (c_2 - c_1A'(0))e_n^2 + \left\{ -\frac{2c_2^2}{c_1} + 2(c_3 + c_2A'(0)) - \frac{1}{2}c_1A''(0) \right\} e_n^3 \\ &\quad + \dots + O(e_n^9), \end{aligned} \quad (2.7)$$

$$\begin{aligned} \frac{f(y_n)}{f(x_n)} &= \left\{ \frac{c_2}{c_1} - A'(0) \right\} e_n + \left\{ -\frac{3c_2^2}{c_1^2} + \frac{2c_3 + 3c_2A'(0)}{c_1} - \frac{A''(0)}{2} \right\} e_n^2 \\ &\quad + \dots + O(e_n^9), \end{aligned} \quad (2.8)$$

and

$$f[x_n, y_n] = c_1 + c_2e_n + \left\{ \frac{c_2^2}{c_1} + c_3 - c_2A'(0) \right\} e_n^2 + \dots + O(e_n^9). \quad (2.9)$$

Using the equations (2.7), (2.9), (2.8) and $A'(0) = 0$, $B(0) = 1$, $B'(0) = 1$, in the second step of (2.1), we can find

$$\begin{aligned} z_n &= \alpha + \frac{c_2(c_1(-2c_3 + c_1A''(0)) - c_2^2(-6 + B''(0)))}{2c_1^3} e_n^4 \\ &\quad + \frac{1}{12c_1^4} (-3c_1(-4c_3 + c_1A''(0))(-2c_3 + c_1A''(0)) + 3c_1c_2^2(-4c_3 + c_1A''(0)) \\ &\quad (-20 + 3B''(0)) + 2c_2^2c_2(-12c_4 + c_1A^{(3)}(0)) + c_2^4(54(-4 + B''(0) - 2B^{(3)})) e_n^5 \\ &\quad + \dots + O(e_n^9). \end{aligned} \quad (2.10)$$

By virtue of the above equation, we have

$$\begin{aligned}
f(z_n) &= \frac{c_2(c_1(-2c_3 + c_1A''(0)) - c_2^2(-6 + B''(0)))}{2c_1^2}e_n^4 \\
&+ \frac{1}{12c_1^3}(-3c_1(-4c_3 + c_1A''(0))(-2c_3 + c_1A''(0)) + 3c_1c_2^2(-4c_3 + c_1A''(0)) \\
&(-20 + 3B''(0)) + 2c_2^2c_2(-12c_4 + c_1A^{(3)}(0)) + c_2^4(54(-4 + B''(0) - 2B^{(3)}(0)))e_n^5 \\
&+ \dots + O(e_n^9). \tag{2.11}
\end{aligned}$$

$$f[y_n, z_n] = c_1 + \frac{c_2^2}{c_1}e_n^2 + \dots + O(e_n^9). \tag{2.12}$$

$$\begin{aligned}
\frac{f(z_n)}{f(y_n)} &= \frac{c_1(-2c_3 + c_1A''(0)) - c_2^2(-6 + B''(0))}{c_1}e_n^2 \\
&+ \frac{1}{6c_1^3}\{3c_1c_2(-4c_3(-6 + B''(0)) + c_1A''(0)(-5 + B''(0))) \\
&+ c_1^2(-12c_4 + c_1A^{(3)}(0)) + c_2^3(-72 + 21B''(0) - B^{(3)}(0))\}e_n^3 \\
&+ \dots + O(e_n^9). \tag{2.13}
\end{aligned}$$

$$\begin{aligned}
\frac{f(z_n)}{f(x_n)} &= \frac{c_2(c_1(-2c_3 + c_1A''(0)) - c_2^2(-6 + B''(0)))}{2c_1^3}e_n^3 \\
&+ \frac{1}{12c_1^4}\{-3c_1^2(-4c_3 + c_1A''(0))(-2c_3 + c_1A''(0)) \\
&+ 3c_1c_2^2(-12c_3(-7 + B''(0)) + c_1A''(0)(-22 + 3B''(0))) \\
&+ 2c_1^2c_2(-12c_4 + c_1A^{(3)}(0)) + c_2^4(60B''(0) - 2(126 + B^{(3)}(0)))\}e_n^3 \\
&+ \dots + O(e_n^9). \tag{2.14}
\end{aligned}$$

Finally, using (2.8), (2.13), (2.14), (2.11), (2.12) and $R(0) = 1 - P(0) - Q(0)$, $P'(0) = 0$, $Q'(0) = 0$, $P''(0) = 2$, $R'(0) = 2$, $A''(0) = 0$, $P^{(3)}(0) = 6B''(0) - 12$ in the last step of (2.1), we get the final error expression which is given by

$$\begin{aligned}
e_{n+1} &= \frac{c_2}{48c_1^7}\{2c_1c_3 + c_2^2(-6 + B''(0))\} \\
&\{12c_1^2c_3^2Q''(0) + 12c_1c_2^2c_3(8 + (-6 + B''(0))Q''(0)) \\
&+ 4c_1^2c_2(-6c_4 + c_1A^{(3)}(0)) + c_2^4(108Q''(0) + 3B''(0) \\
&(8 + (-12 + B''(0))Q''(0)) - 8(9 + B^{(3)}(0) + P^{(4)}(0)))\}e_n^8 + O(e_n^9). \tag{2.15}
\end{aligned}$$

Thus, theorem is proved. \square

Particular Case:

Let

$$\begin{aligned} A(t_1) &= 1 + \alpha t_1^3, \\ B(t_2) &= 1 + t_2 + \beta t_2^2, \\ C(t_2) &= t_2^2 + 2(\beta - 1) t_2^3, \\ D(t_3) &= \gamma t_3^2, \\ E(t_4) &= 1 + 2 t_4 + \delta t_4^2, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, then the method becomes

$$\begin{aligned} y_n &= x_n - \{1 + \alpha t_1^3\} \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \{1 + t_2 + \beta t_2^2\} \frac{f(y_n)}{f[x_n, y_n]}, \\ x_{n+1} &= z_n - \{\{t_2^2 + 2(\beta - 1) t_2^3\} + \{\gamma t_3^2\} + \{1 + 2 t_4 + \delta t_4^2\}\} \cdot \frac{f(z_n)}{f[y_n, z_n]}. \end{aligned} \quad (2.16)$$

Clearly, this method is four-parametric where $t_1 = \frac{f(x_n)}{f'(x_n)}$, $t_2 = \frac{f(y_n)}{f(x_n)}$, $t_3 = \frac{f(z_n)}{f(y_n)}$ and $t_4 = \frac{f(z_n)}{f(x_n)}$. Then its error expression becomes

$$\begin{aligned} e_{n+1} &= \frac{c_2}{c_1^7} \{c_1 c_3 + c_2^2(-3 + \beta)\} \\ &\quad \{ \alpha c_1^2 c_2 + (-3 + 2\beta + (-3 + \beta^2)\gamma) c_2^4 \\ &\quad + 2(2 + (-3 + \beta)\gamma) c_1 c_2^2 c_3 + c_1^2 (\gamma c_3^2 - c_2 c_4) \} e_n^8 + O(e_n^9). \end{aligned} \quad (2.17)$$

Remark: By taking different values of α, β, γ and δ one can get a number of eighth-order iterative methods. Our class of three-step method requires four evaluations (one derivative and three function) and has the order of convergence eight. Therefore our class is of optimal order and support the Kung-Traub conjecture for $n = 4$. Clearly its efficiency index is $8^{\frac{1}{4}} \approx 1.682$ which is more than efficiency index $7^{\frac{1}{4}} \approx 1.626$ of method (1.2).

3. DERIVATIVE-FREE SCHEME

In the real world problems of science and engineering sometimes the derivative of the function is not easy to calculate or time consuming. To

overcome this problem, in recent days many researches have focused to develop derivative-free methods to solve real world problems e.g. [15], [16] and [17]. In this section, we give the derivative free version of the method of previous section.

To do so, we replace $f'(x_n) \approx f[w_n, x_n]$ in the equation (2.1) where $w_n = x_n + f(x_n)$, then the method becomes

$$\begin{aligned} y_n &= x_n - A(t_1) \cdot \frac{f(x_n)}{f[w_n, x_n]}, \\ z_n &= y_n - B(t_2) \cdot \frac{f(y_n)}{f[x_n, y_n]}, \\ x_{n+1} &= z_n - \{P(t_2) + Q(t_3) + R(t_4)\} \cdot \frac{f(z_n)}{f[y_n, z_n]}, \end{aligned} \quad (3.1)$$

and it can be seen that the error equation under the same conditions on weight functions as of theorem (2.1) is given by

$$\begin{aligned} e_{n+1} &= \frac{(1 + c_1)c_2^4(-2 + c_1(-2 + Q''[0]))e_n^5}{2c_1^2} \\ &\quad + O(e_n^6), \end{aligned} \quad (3.2)$$

which show fifth-order of convergence. Again to maintain its order of convergence we consider, $w_n = x_n + f(x_n)^2$, then we see that the order of convergence is seven and its error expression is given by

$$\begin{aligned} e_{n+1} &= -\frac{\left(c_2^3(2c_1^3c_2 + 2c_1c_3 + c_2^2(-6 + B''[0]))\right)e_n^7}{2c_1^3} \\ &\quad + O(e_n^8), \end{aligned}$$

which also does not meet with our aim. But if we put $w_n = x_n + f(x_n)^3$ then its error equation (under the same conditions on weight functions as of theorem (2.1)).

$$\begin{aligned} e_{n+1} &= \frac{1}{48c_1^7}c_2(2c_1c_3 + c_2^2(-6 + B''[0]))\left(-24c_1^5c_2^2 + 12c_1^2(-2c_2c_4 + c_3^2Q''[0])\right. \\ &\quad + 12c_1c_2^2c_3(8 + (-6 + B''[0])Q''[0]) + 4c_1^3c_2A^{(3)}[0] + c_2^4(108Q''[0] \\ &\quad + 3B''[0](8 + (-12 + B''[0])Q''[0]) - 8(9 + B^{(3)}[0]) + P^{(4)}[0])\left.)\right)e_n^8 \\ &\quad + O(e_n^9). \end{aligned} \quad (3.3)$$

Thus the method preserves its order of convergence for $w_n = x_n + f(x_n)^3$. In fact if we put $w_n = x_n + \alpha \cdot (f(x_n))^n$, $n \geq 3$, where $\alpha \neq 0 \in \mathbb{R}$ in the scheme (3.1) then it gives eighth-order of convergence.

4. RESULTS AND DISCUSSION

This section deals with the numerical comparisons of the proposed method (2.16) with $\alpha = \gamma = 0$, $\beta = 3$, $\delta = 1$. In order to check the effectiveness of the proposed iterative method we have considered seven test nonlinear functions which are taken from [3]. The test nonlinear functions and their roots are listed in Table-1. In recent days, higher-order methods are very important because numerical applications use high precision computations. Due to this reason all the computations reported have been performed in the programming package *MATHEMATICA* 8 using 1000 digits floating point arithmetic using "*SetAccuracy*" command. The results of comparisons are given in Table 2 and Table 3. The computer characteristics during numerical calculations are Microsoft Windows 8 Intel(R) Core(TM) i5-3210M CPU@ 2.50 GHz with 4.00 GB of RAM, 64-bit Operating System throughout this paper. Here we compare performances of our new eighth-order method (2.16) (*OM8*) with the methods of (34) ($M_{8,1}$), (35) ($M_{8,2}$) of [14]; methods NM2 ($M_{8,3}$), NM3 ($M_{8,4}$) of [2] and methods (11) ($M_{8,5}$) (15) ($M_{8,6}$) of [3]. Table 2 represents the value of $|f(x_n)|$ calculated for total number of function evaluations 12 (TNFE-12) for each scheme. It can be observed from Table 2 in almost cases our method *OM8* is superior than other methods. Table 3 exhibits the number of iteration and total number of function evaluation using the stopping criteria $|f(x_{n+1})| < \epsilon$ where $\epsilon = 10^{-50}$. From Table 3, we observe that *OM8* takes at least equal or less number of iterations for different initial guesses.

TABLE 1. Functions and their roots.

$f(x)$	α
$f_1(x) = 10xe^{-x^2} - 1$	$\alpha_1 \approx 1.67963...$
$f_2(x) = x^5 + x^4 + 4x^2 - 15$	$\alpha_2 \approx 1.34742...$
$f_3(x) = xe^{x^2} - (\text{Sin}x)^2 + 3\text{Cos}x + 5$	$\alpha_3 \approx -1.20764...$
$f_4(x) = x^4 + \text{Sin}(\frac{\pi}{x^2}) - 5$	$\alpha_4 = \sqrt{2}$
$f_5(x) = x^2e^x - \text{Sin}x$	$\alpha_5 = 0$
$f_6(x) = (\text{Sin}x - \frac{\sqrt{2}}{2})^2(x + 1)$	$\alpha_6 = -1$
$f_7(x) = \text{Sin}3x + x\text{Cos}x$	$\alpha_7 \approx 1.19776...$

TABLE 2. Comparison of absolute value of the functions by different methods after third iteration (TNFE-12).

$ f $	Guess	$M_{8,1}$	$M_{8,2}$	$M_{8,3}$	$M_{8,4}$	$M_{8,5}$	$M_{8,6}$	OM8
$ f_1 $	1.72	0.2e-617	0.2e-617	0.2e-636	0.2e-602	0.1e-660	0.2e-654	0.4e-688
	1.5	0.1e-357	0.7e-358	0.7e-365	0.3e-346	0.2e-375	0.8e-361	0.2e-448
	1.7	0.5e-796	0.5e-796	0.1e-794	0.2e-762	0.4e-828	0.5e-809	0.4e-866
	1.1	0.8e-259	0.7e-257	0.3e-179	0.6e-175	0.1e-205	0.2e-204	0.6e-259
$ f_2 $	1.1	NC	0.6e-52	0.1e-177	0.9e-116	0.7e-165	0.5e-299	0.3e-127
	1.8	0.3e-148	0.2e-149	0.5e-194	0.2e-175	0.1e-195	0.4e-187	0.1e-225
	1.5	0.6e-347	0.6e-347	0.2e-377	0.1e-348	0.9e-437	0.1e-390	0.4e-436
	2.0	0.5e-97	0.6e-99	0.3e-148	0.1e-133	0.3e-134	0.4e-132	0.1e-150
$ f_3 $	-1.1	0.5e-234	0.1e-235	0.4e-337	0.8e-285	0.1e-325	0.6e-433	0.1e-301
	-1.5	0.5e-124	0.7e-125	0.2e-182	0.2e-161	0.2e-253	0.2e-205	0.2e-254
	-1.0	div.	0.2e-45	0.1e-173	0.2e-107	0.3e-158	0.3e-254	0.9e-116
	-1.3	0.1e-383	0.1e-383	0.6e-404	0.2e-370	0.2e-411	0.3e-460	0.1e-468
$ f_4 $	1.0	0.9e-255	0.1e-251	0.2e-226	0.3e-223	0.2e-199	0.5e-227	0.2e-262
	1.6	0.8e-338	0.7e-338	0.9e-356	0.2e-329	0.3e-370	0.5e-430	0.1e-441
	1.5	0.2e-508	0.2e-508	0.1e-519	0.1e-487	0.9e-526	0.4e-560	0.3e-532
	2.1	0.1e-105	0.2e-107	0.1e-146	0.9e-134	0.1e-144	0.1e-143	0.3e-159
$ f_5 $	0.1	0.1e-272	0.3e-273	0.1e-340	0.4e-303	0.6e-349	0.1e-338	0.1e-364
	0.5	0.5e-264	0.8e-265	0.1e-346	0.3e-301	0.2e-342	0.7e-382	0.5e-339
	-0.1	0.4e-475	0.1e-475	0.2e-470	0.1e-455	0.3e-441	0.2e-436	0.4e-485
	-0.5	0.1e-270	0.3e-270	0.8e-237	0.2e-233	0.1e-235	0.3e-233	0.2e-277
$ f_6 $	-0.8	0.3e-158	0.1e-162	0.4e-258	0.9e-214	0.4e-248	0.6e-288	0.3e-254
	-1.2	0.3e-422	0.4e-423	0.7e-398	0.9e-387	0.3e-385	0.1e-380	0.2e-435
	-0.9	0.1e-425	0.1e-425	0.1e-456	0.1e-424	0.4e-500	0.4e-457	0.6e-526
	-1.5	0.1e-324	0.8e-324	0.7e-262	0.1e-258	0.7e-273	0.1e-272	0.3e-336
$ f_7 $	1.0	0.1e-446	0.2e-448	0.1e-374	0.1e-371	0.2e-378	0.7e-374	0.2e-505
	0.8	0.6e-81	NC	0.3e-130	0.4e-114	0.7e-159	0.1e-116	0.6e-150
	1.8	NC	NC	0.2e-38	0.3e-20	NC	NC	0.6e-29
	0.3	0.1e-226	0.3e-235	0.8e-244	0.7e-215	0.7e-277	0.1e-278	0.2e-416

Here div.= Divergent , I= Indeterminate , NC= Not convergent.

TABLE 3. Comparison of number of iterations and total number of function evaluations (TNFE).

[illegible]

5. CONCLUSION AND FUTURE WORK

In this article, we have contributed a new efficient family of eight-order iterative methods to find simple roots of a nonlinear equation by accelerating the order of convergence and efficiency index of well existing seventh-order iterative method of [1] without using more function evaluations for finding simple roots of nonlinear equations. Our family requires three function and one derivative evaluations and thus agrees with the conjecture of Kung-Traub for the case $n = 4$ (i.e. optimal). An approach to make proposed method free from derivative has also discussed here. Numerical comparisons also witness the efficiency of new method. Therefore, we can conclude that the new family is efficient and give at least equal or better performance over some other eight-order methods. Using the technique of [17] other existing methods having derivatives can be made free from derivatives.

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